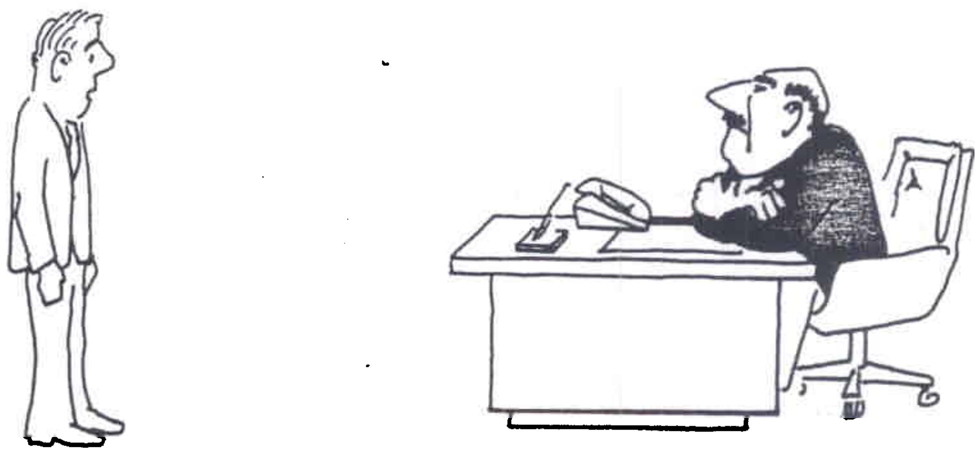
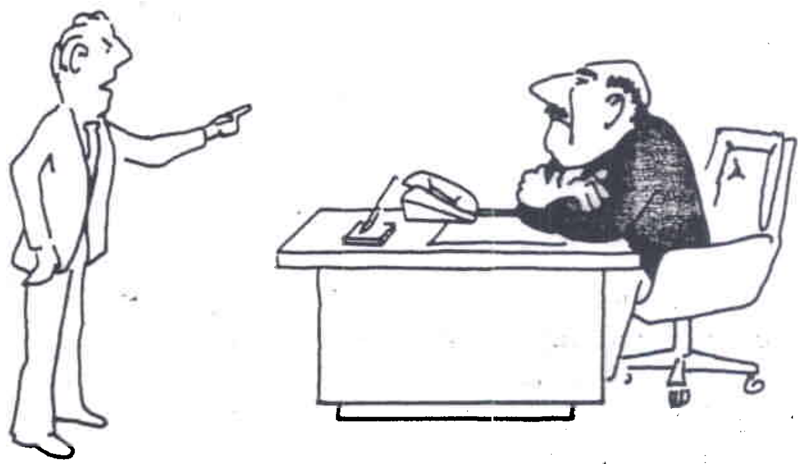


"I can't find an efficient algorithm, I guess I'm just too dumb."



"I can't find an efficient algorithm, because no such algorithm is possible!"

"I can't find an efficient algorithm, but neither can all these famous people."

Sat $\in NP$: proof = satisfying assignment

Graph coloring $\in NP$: proof = coloring

k-clique, k-vertex cover, k-independent set $\in NP$

Tautology $\in$ co-NP ("no" instances have a proof)

NP complete:

1) in NP

2) <u>every</u> problem in NP reducible to it.

Transitivity of p-time reduction implies

NP complete iff

1) in NP

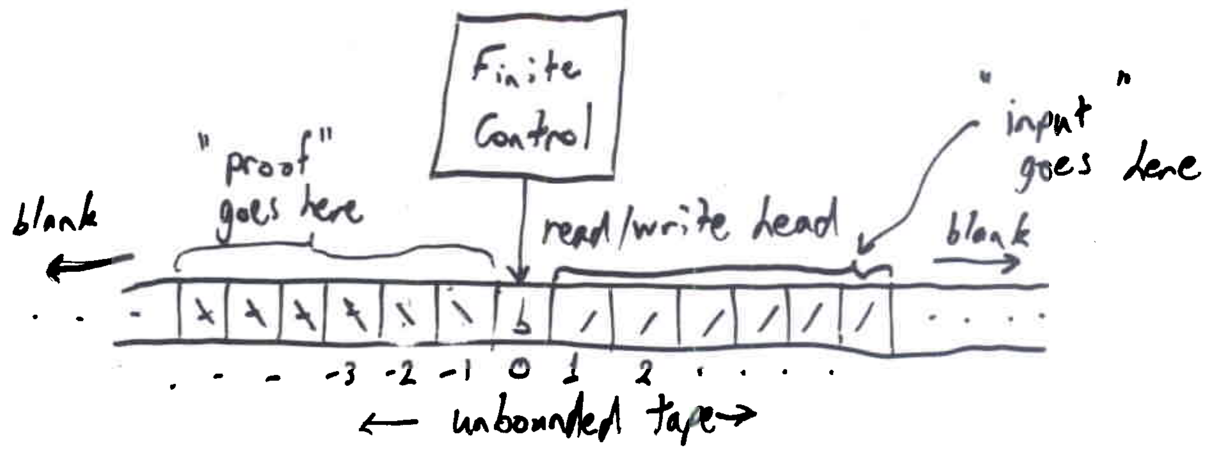
2') <u>some</u> NP-complete problem reducible to it.

We need one NP-complete problem to get started.

Cook-Levin Theorem: Sat is NP-complete.

Given any p-time verifier, construct (in p-time) an instance of Sat s.t. verifier answers "yes" iff formula is satisfiable.

Verifier: Turing Machine



In one step, machine can write a symbol, move head one position, change state.

What to do is based on state, symbol read.

Fixed # of states, fixed # of tape symbols, including blank; start state, "yes" state, ("no" state)

Explicitly given polynomial time bound $p(n)$.

Input (of size n) is a "yes" instance iff

for some "proof" and given input, the machine

reaches "yes" state within $p(n)$ steps from

start state.

Must construct a formula that is satisfiable

iff this happens.

Note: input is specified, proof is not (non deterministic

part)

Proof can't exceed length $p(n)$: machine can't get

farther in $p(n)$ steps.

Can assume machine loops in "yes" state: if

ever in "yes", will be in "yes" at step $p(n)$.

States: 1, ..., y   1 = start, y = yes

Symbols: 1, ..., z   1 = blank

Tape cells, $-p(n), \ldots, 0, \ldots, p(n)$

Time: $0, 1, \ldots, p(n)$

Variables for formula:

h_{it}: true if head on tape cell i at time t
$-p(n) \le i \le p(n), \quad 0 \le t \le p(n)$

s_{jt}: true if state j at time t
$1 \le j \le y, \quad 0 \le t \le p(n)$

c_{ikt}: true if tape cell i holds symbol k at time t
$-p(n) \le i \le p(n), \quad 1 \le k \le z, \quad 0 \le t \le p(n)$

What does the formula need to say?

At most one state, head position, and symbol per cell at each time:

$(\bar{h}_{it} \vee \bar{h}_{i't})$ $i \neq i'$, all t

$(\bar{s}_{jt} \vee \bar{s}_{j't})$ $j \neq j'$, all t

$(\bar{c}_{ikt} \vee \bar{c}_{ik't})$ $k \neq k'$, all i, all t

Correct initial state, head position, and tape contents:

$$h_{00} \wedge s_{10} \wedge c_{010} \wedge c_{1k_10} \wedge c_{2k_20} \wedge \cdots \wedge c_{nk_n0}$$

$$\wedge\ c_{(n+1)10} \wedge \cdots \wedge c_{p(n)10}$$

Input is $k_1 k_2 \ldots k_n$, rest of right side of tape is blank

Correct final state: $s_{y\,p(n)}$

Correct transitions:

E.g. if machine in state j reads k, it then writes k', moves head right, and changes to state j':

$$s_{jt} \wedge h_{it} \wedge c_{ikt} \supset s_{j'\,t+1} \wedge h_{i+1\,t+1} \wedge c_{ik'\,t+1}$$

($\supset$ = "implies") (for each i[illegible])

$$h_{it} \wedge c_{i'kt} \supset c_{i'k\,t+1}$$ (for $i \neq i'$, each k, t)

(unread tape cells are unaffected)

CNF?

Any proof that gives a "yes" execution gives a satisfying assignment, and vice-versa.

Conclusion: SAT is NP-complete

(and k-coloring, k-clique, k-independent set, k-vertex cover)

$$(x \wedge y \wedge z) \supset (a \wedge b \wedge c)$$

$$\Rightarrow$$

$$((x \wedge y \wedge z) \supset a)$$
$$\wedge$$
$$((x \wedge y \wedge z) \supset b)$$
$$\wedge$$
$$((x \wedge y \wedge z) \supset c)$$

$$\Rightarrow$$

$$(\bar{x} \vee \bar{y} \vee \bar{z} \vee a)$$
$$\wedge$$
$$(\bar{x} \vee \bar{y} \vee \bar{z} \vee b)$$
$$\wedge$$
$$(\bar{x} \vee \bar{y} \vee \bar{z} \vee c)$$

Subset Sum is NP-complete

Given m integers, and a target k, is there a subset that sums to exactly k?

$\{2, 5, 6, 8, 9, 12\}$ $k = 31$

yes: 5, 6, 8, 12 n = # digits

(no for $k = 30$)

In NP: subset is proof (verifiable in p-time)

Some NPC problem reducible to subset sum

reduce 3-CNF sat to subset sum

Write numbers base 10

$$(x \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee y \vee z) \wedge (y \vee \bar{z})$$

C_1 $\quad$ C_2 $\quad$ C_3

	x	y	z	C_1	C_2	C_3	
x	1	0	0	1	0	0	
$\bar{x}$	1	0	0	0	1	0	
y	0	1	0	0	1	1	← y makes C_2, C_3 true
$\bar{y}$	0	1	0	1	0	0	
z	0	0	1	0	1	0	
$\bar{z}$	0	0	1	1	0	1	← $\bar{z}$ makes C_1, C_2 true
Dummies to get clause columns to sum to 4	0	0	0	1	0	0	
	0	0	0	2	0	0	
	0	0	0	0	1	0	
	0	0	0	0	2	0	
	0	0	0	0	0	1	
	0	0	0	0	0	2	
	1	1	1	4	4	4	= k Required sum

Interpret each row as a base 10 number.

Subset sum has a solution iff formula is satisfiable.